Rossmoyne Senior High School

Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed 65% (97 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

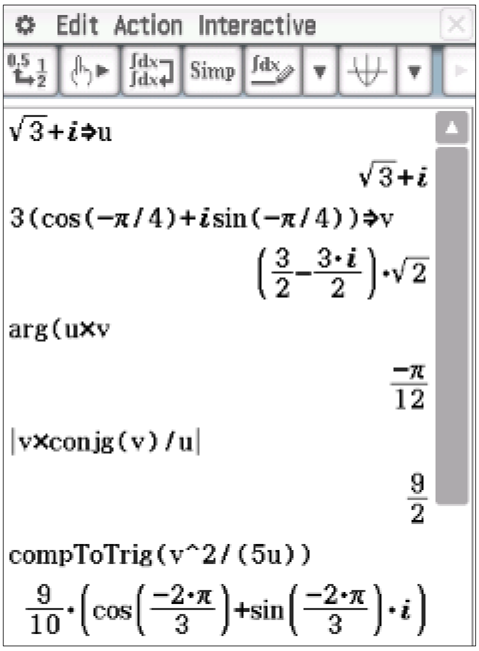
Working time: 100 minutes.

Question 9 (6 marks)

Two complex numbers are $u = \sqrt{3} + i$ and $v = 3\ \mathrm{cis}\left(-\frac{\pi}{4}\right)$.

(a) Determine the argument of uv. (2 marks)

Solution
$\arg(u) = \frac{\pi}{6}$ $\arg(uv) = \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$
Specific behaviours
✓ argument of u ✓ argument



(b) Simplify $|v \times \bar{v} \times u^{-1}|$. (2 marks)

Solution
$|v \times \bar{v}| \times \left|\frac{1}{u}\right| = |v|^2 \times \left|\frac{1}{u}\right| = 9 \times \frac{1}{2}$ $= \frac{9}{2}$
Specific behaviours
✓ evaluates $|v \times \bar{v}|$ ✓ correct magnitude

(c) Determine z in polar form if $5zu = v^2$. (2 marks)

Solution
$z = \frac{v^2}{5u} = \frac{1}{5} \times \frac{9\ \mathrm{cis}\left(-\frac{\pi}{2}\right)}{2\ \mathrm{cis}\left(\frac{\pi}{6}\right)}$ $z = \frac{9}{10}\mathrm{cis}\left(-\frac{2\pi}{3}\right)$
Specific behaviours
✓ indicates v^2 in polar form   ✓ simplifies z

Question 10 **(5 marks)**

(a) The vector equation of a curve is given by $\mathbf{r}(\mu) = (\mu^2 + 3)\mathbf{i} + (\mu - 5)\mathbf{j}$. Determine the corresponding Cartesian equation for the curve. (2 marks)

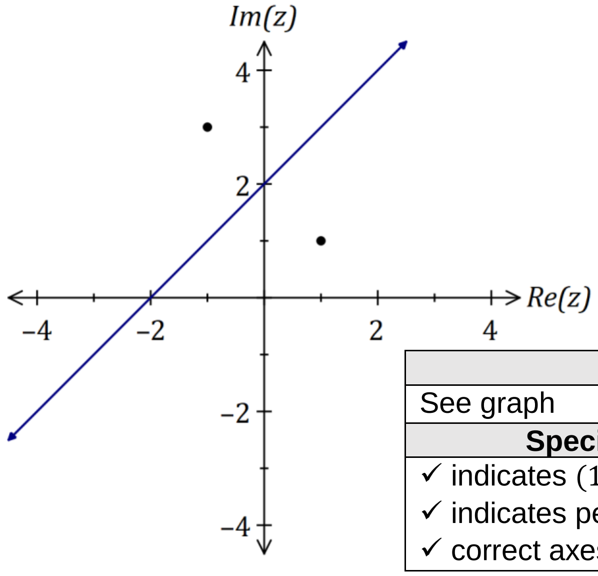
Solution
$y = \mu - 5 \Rightarrow \mu = y + 5$ $x = \mu^2 + 3$ $x = (y + 5)^2 + 3$
Specific behaviours
✓ express μ in terms of y ✓ Cartesian equation

(b) A sphere has Cartesian equation $x^2 + y^2 + z^2 + 8x - 12y + 2z = 0$. Determine the vector equation of the sphere. (3 marks)

Solution
$(x + 4)^2 + (y - 6)^2 + (z + 1)^2 = 4^2 + 6^2 + 1^2 = 53$ $\left\| \mathbf{r} - \begin{pmatrix} -4 \\ 6 \\ -1 \end{pmatrix} \right\| = \sqrt{53}$
Specific behaviours
✓ completes squares ✓ correct radius ✓ correct vector form

Question 11 **(9 marks)**

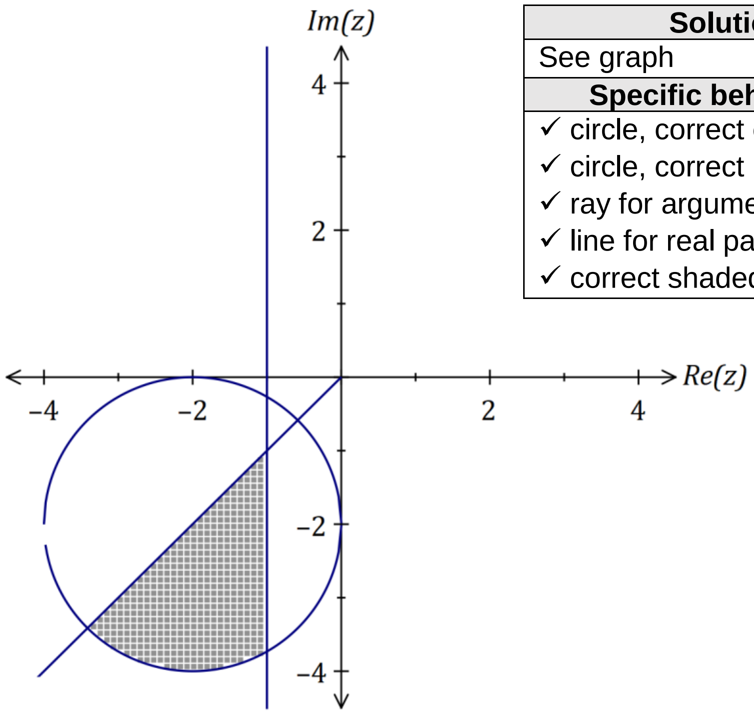
(a) On the Argand plane below, sketch the locus of $|z-1-i| = |z+1-3i|$, where z is a complex number. (3 marks)



Solution
See graph
Specific behaviours
✓ indicates $(1, 1)$ and $(-1, 3)$
✓ indicates perpendicular bisector
✓ correct axes intercepts

(b) Consider the three inequalities $|z+2+2i| \le 2$, $\arg(z) \ge -\dfrac{3\pi}{4}$ and $\text{Re}(z) \le -1$.

(i) On the Argand plane below, shade the region that represents the complex numbers satisfying these inequalities. (5 marks)



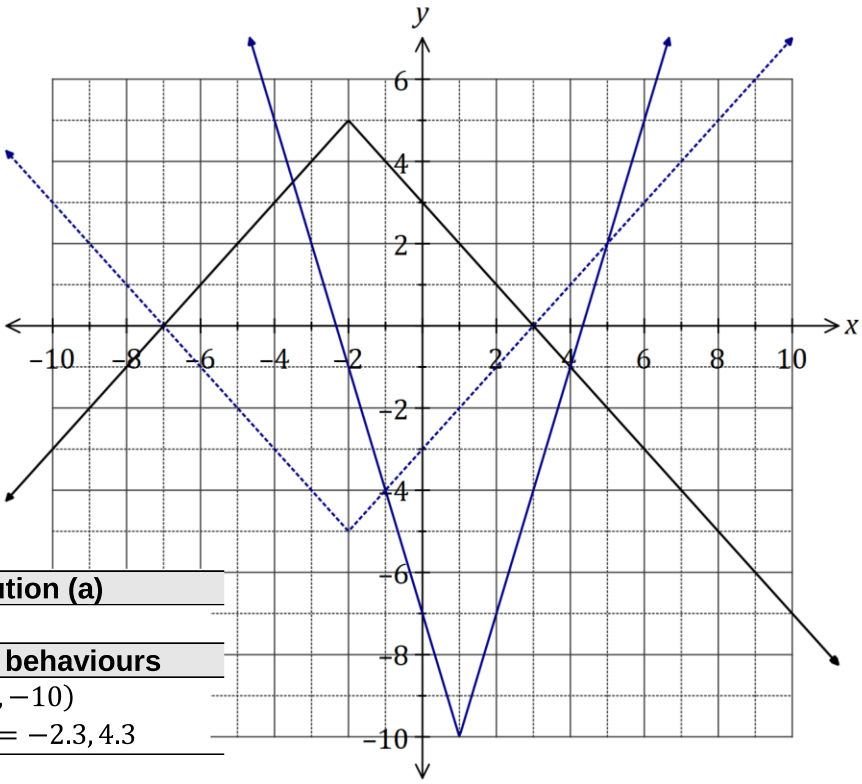
Solution
See graph
Specific behaviours
✓ circle, correct centre
✓ circle, correct radius
✓ ray for argument
✓ line for real part
✓ correct shaded region

(ii) Determine the maximum possible value of $|z|$ within the shaded region. (1 mark)

Solution
$2\sqrt{2}+2$
Specific behaviours
✓ correct value

Question 12 **(7 marks)**

The graph of $y = f(x)$ is shown below, where $f(x) = a|x + b| + c$, where a, b and c are constants.



Solution (a)
See graph
Specific behaviours
✓ cusp at $(1, -10)$
✓ roots at $x = -2.3, 4.3$

(a) Add the graph of $y = g(x)$ to the axes above, where $g(x) = 3|x - 1| - 10$. (2 marks)

(b) Determine the values of a, b and c. (3 marks)

Solution
Slopes: $m \pm 1 \Rightarrow a = -1$ From cusp: $b = 2$ and $c = 5$
Specific behaviours
✓ correct value of a ✓ correct value of b ✓ correct value of c

(c) Using your graph, or otherwise, solve $f(x) + g(x) = 0$. (2 marks)

Solution
Using reflection of $y = f(x)$ in $y = 0$, graphs intersect when $x = -1, x = 5$.
Specific behaviours
✓ reflects $f(x)$ ✓ both solutions

Question 13 **(7 marks)**

A particle, with initial velocity vector $(10, 3, -15)$ ms^{-1}, experiences a constant acceleration for 16 seconds. The velocity vector of the particle at the end of the 16 seconds is $(34, -37, -31)$ ms^{-1}.

(a) Determine the magnitude of the acceleration. (3 marks)

Solution
$\Delta\mathbf{v} = \begin{pmatrix} 34 \\ -37 \\ -31 \end{pmatrix} - \begin{pmatrix} 10 \\ 3 \\ -15 \end{pmatrix} = \begin{pmatrix} 24 \\ -40 \\ -16 \end{pmatrix}$ $\mathbf{a} = \dfrac{\Delta\mathbf{v}}{16} = \begin{pmatrix} 1.5 \\ -2.5 \\ -1 \end{pmatrix}$ $\lvert\mathbf{a}\rvert = \dfrac{\sqrt{38}}{2} \approx 3.082 \text{ ms}^{-2}$
Specific behaviours
✓ change in velocity   ✓ acceleration vector   ✓ magnitude

(b) Calculate the change in displacement of the particle over the 16 seconds. (4 marks)

Solution
$\mathbf{v} = \begin{pmatrix} 10 \\ 3 \\ -15 \end{pmatrix} + t\begin{pmatrix} 1.5 \\ -2.5 \\ -1 \end{pmatrix}$ $\Delta\mathbf{r} = \int_0^{16} \mathbf{v}\, dt$ $\Delta\mathbf{r} = \left[t\begin{pmatrix} 10 \\ 3 \\ -15 \end{pmatrix} + \dfrac{t^2}{2}\begin{pmatrix} 1.5 \\ -2.5 \\ -1 \end{pmatrix} \right]_0^{16}$ $\Delta\mathbf{r} = \begin{pmatrix} 352 \\ -272 \\ -368 \end{pmatrix}$
Specific behaviours
✓ velocity vector   ✓ indicates use of integration   ✓ correct displacement vector   ✓ change in displacement

Question 14 **(8 marks)**

The position vectors of bodies L and M at times λ and μ are given by

$$\mathbf{r}_L = 10\mathbf{i} - \mathbf{j} + 6\mathbf{k} + \lambda(-2\mathbf{i} + a\mathbf{j} + b\mathbf{k})$$

and

$$\mathbf{r}_M = 3\mathbf{i} - 11\mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

where a and b are constants, times are in seconds and distances are in metres.

(a) Given that the paths of L and M intersect, show that $4a - b + 10 = 0$. (4 marks)

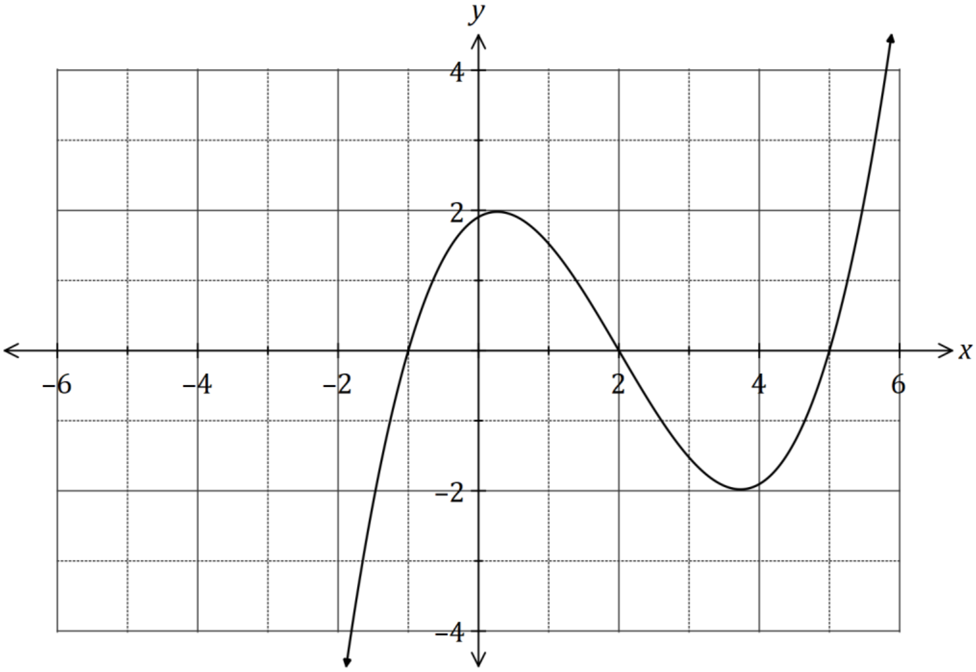
Solution
From i coefficients: $10 - 2\lambda = 3 + \mu \Rightarrow \mu = 7 - 2\lambda$ From j coefficients: $a\lambda - 1 = 2\mu - 11 \Rightarrow a\lambda = 14 - 4\lambda - 10 \Rightarrow \lambda = \frac{4}{a+4}$ From k coefficients: $6 + b\lambda = 1 + 3\mu \Rightarrow b\lambda = -5 + 21 - 6\lambda \Rightarrow \lambda = \frac{16}{b+6}$ Hence $4a + 16 = b + 6 \Rightarrow 4a - b + 10 = 0$
Specific behaviours
✓ relates μ and λ from k coefficients ✓ uses j coefficients to express μ or λ in terms of b ✓ uses i coefficients to express μ or λ in terms of a ✓ equates expressions and simplifies

(b) Given that the paths of L and M are also perpendicular, determine the values of a and b, and the position vector of the point of intersection of the paths. (4 marks)

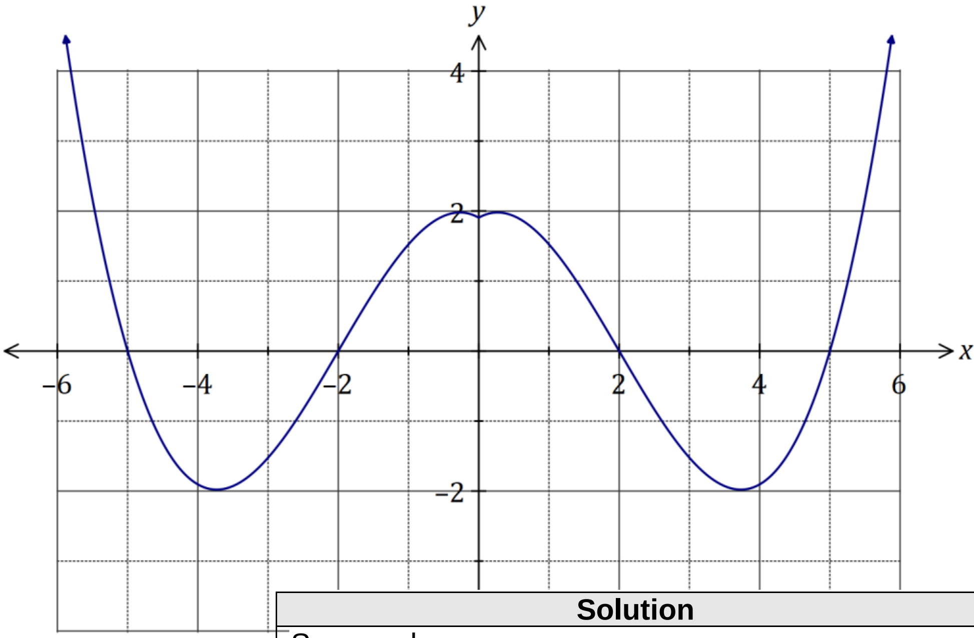
Solution
$(-2\mathbf{i} + a\mathbf{j} + b\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = -2 + 2a + 3b = 0$ $-2 + 2a + 3b = 0 \;\;\&\;\; 4a - b + 10 = 0$ $a = -2, \qquad b = 2$ $\lambda = 4 \div 2 = 2$ $\mathbf{r} = 10\mathbf{i} - \mathbf{j} + 6\mathbf{k} + 2(-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$
Specific behaviours
✓ equates scalar product of directions to 0   ✓ solves for a and b ✓ determines λ ✓ states position vector of point of intersection

Question 15 **(8 marks)**

The graph of $y = f(x)$ is shown below.

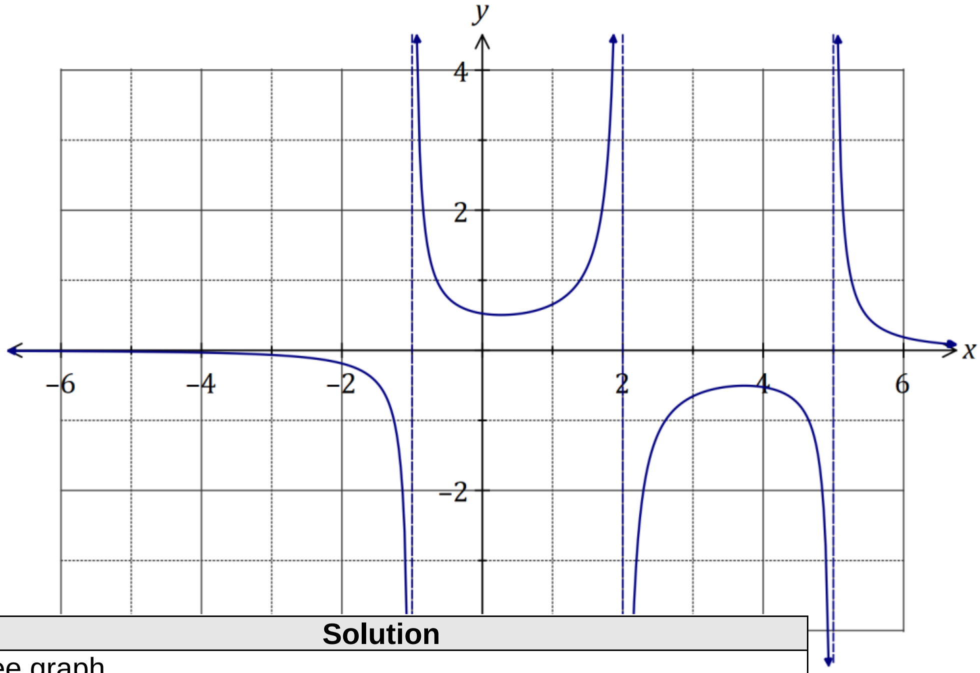


(a) Sketch the graph of $y = f(|x|)$ on the axes below. (2 marks)



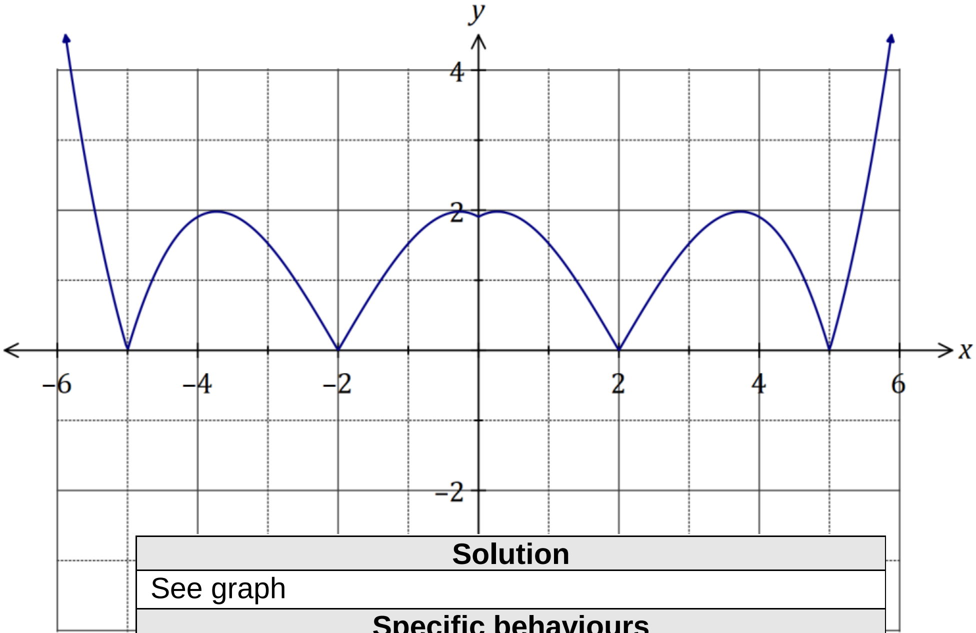
Solution
See graph
Specific behaviours
✓ indicates roots and slight cusp close to $(0, 1.9)$ ✓ indicates symmetry about $x = 0$

(b) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below. (4 marks)



Solution
See graph
Specific behaviours
✓ indicates vertical asymptotes at $x = -1, 2$ and 5
✓ indicates $y \to 0$ for $\lvert x \rvert \to \infty$
✓ indicates turning points close to $(3.7, -0.5)$ and $(0.3, 0.5)$
✓ indicates correct curvature between asymptotes

(c) Sketch the graph of $y = \lvert f(\lvert x \rvert)\rvert$ on the axes below. (2 marks)



Solution
See graph
Specific behaviours
✓ indicates symmetry about $x = 0$
✓ reflects parts of graph (a) below $y = 0$ above axis

Question 16 **(8 marks)**

(a) Let $r\,\text{cis}\,\theta$ be a point in the complex plane. Determine, in terms of r and θ, the polar form of this point after it is rotated by $\frac{\pi}{4}$ about the origin and then reflected in the real axis.

(2 marks)

Solution
$z \to z_1 = r\,\text{cis}\left(\theta + \frac{\pi}{4}\right)$ $z_1 \to \overline{z_1} = r\,\text{cis}\left(-\theta - \frac{\pi}{4}\right)$
Specific behaviours
✓ rotation   ✓ conjugate

(b) Let $f(w) = -i\overline{w} + 1 + i$.

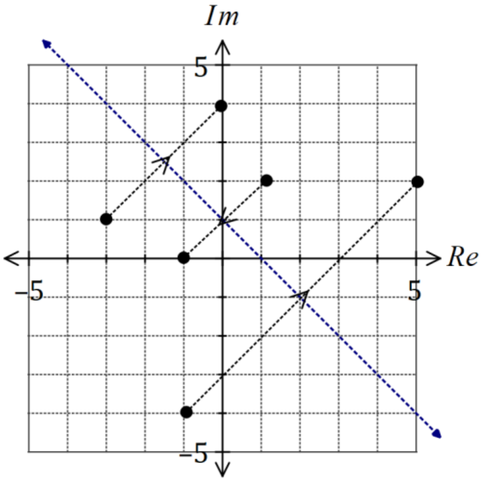
(i) Complete the following table. (3 marks)

w	$1 + 2i$	$-3 + i$	$-1 - 4i$
$f(w)$	-1	$4i$	$5 + 2i$

Solution
See table
Specific behaviours
✓✓✓ each point

(ii) Sketch each point, w, and join it with a dotted line to its image, $f(w)$, on the diagram below. (1 mark)

Solution
See diagram
Specific behaviours
✓ plots points



(iii) Describe the geometric transformation that $f(w)$ represents. (2 marks)

Solution
Reflection in the line $\text{Re}(z) + \text{Im}(z) = 1$
Specific behaviours
✓ reflection   ✓ line of reflection

Question 17 **(8 marks)**

The velocity vector of a small body at time t seconds is $\mathbf{v}(t) = 4\sin(4t)\,\mathbf{i} + 3\cos(4t)\,\mathbf{j}$ ms^{-1}. Initially, the body has position vector $2\mathbf{i} + 3\mathbf{j}$.

(a) Determine the acceleration vector for the body when $t = \frac{5\pi}{16}$. (2 marks)

Solution
$\mathbf{a}(t) = 16\cos(4t)\,\mathbf{i} - 12\sin(4t)\,\mathbf{j}$ $\mathbf{a}\left(\frac{5\pi}{16}\right) = (-8\sqrt{2}, 6\sqrt{2})$
Specific behaviours
✓ acceleration vector   ✓ acceleration

(b) Show that the minimum speed of the body is 3 ms^{-1}. (3 marks)

Solution
$s^2 = 16\sin^2(4t) + 9\cos^2(4t)$ $= 7\sin^2(4t) + 9\sin^2(4t) + 9\cos^2(4t)$ $= 7\sin^2(4t) + 9$ s^2 is min when $\sin^2(4t) = 0$ $s_{min} = \sqrt{0+9} = 3$ m/s
Specific behaviours
✓ expression for s or s^2 ✓ simplifies expression   ✓ indicates minimum with justification

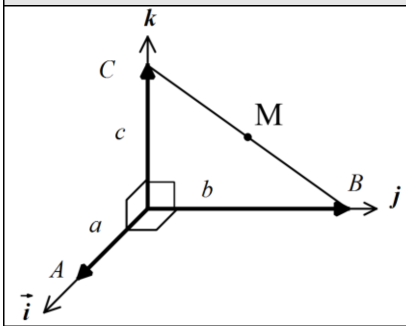
(c) Determine the distance the body travels between $t = 0$ and the first instant after this time that the body returns to its initial position, rounding your answer to the nearest cm. (3 marks)

Solution
Period of motion is $\frac{2\pi}{4} = \frac{\pi}{2}$ $d = \int_0^{\frac{\pi}{2}} \sqrt{16\sin^2(4t) + 9\cos^2(4t)}\,dt$ $d = 5.53$ m
Specific behaviours
✓ period   ✓ indicates valid integral   ✓ distance (*no rounding penalty*)

Question 18 **(8 marks)**

Points A, B and C have position vectors $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$ respectively, where a, b and c are non-zero, real constants. Point M is the midpoint of B and C.

(a) Sketch a vector diagram representing this situation. (1 mark)

Solution	Specific behaviours
	✓ appropriate diagram with points and vectors indicated

(b) Use a vector method to prove that $\overrightarrow{AM}$ is perpendicular to $\overrightarrow{BC}$ when $|\overrightarrow{OB}| = |\overrightarrow{OC}|$. (7 marks)

Solution

$$\overrightarrow{BC} = -b\mathbf{j} + c\mathbf{k}$$

$$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AO} + \overrightarrow{OB} + \frac{1}{2}\left(\overrightarrow{BC}\right)\\ &= -a\mathbf{i} + b\mathbf{j} + \frac{1}{2}(-b\mathbf{j} + c\mathbf{k})\\ &= -a\mathbf{i} + \frac{1}{2}b\mathbf{j} + \frac{1}{2}c\mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} \cdot \overrightarrow{AM} &= (-b\mathbf{j} + c\mathbf{k}) \cdot \left(-a\mathbf{i} + \frac{1}{2}b\mathbf{j} + \frac{1}{2}c\mathbf{k}\right)\\ &= (-b\mathbf{j}) \cdot \left(\frac{1}{2}b\mathbf{j}\right) + (c\mathbf{k}) \cdot \left(\frac{1}{2}c\mathbf{k}\right)\\ &= \frac{1}{2}(c^2 - b^2)\\ &= 0 \text{ as } \left|\overrightarrow{OB}\right| = \left|\overrightarrow{OC}\right|\end{aligned}$$

Hence $\overrightarrow{BC} \perp \overrightarrow{AM}$ as the scalar product of non-zero vectors is only zero when vectors are perpendicular.

Specific behaviours

✓ vector for $\overrightarrow{BC}$
✓ indicates method for vector $\overrightarrow{AM}$
✓ simplifies vector for $\overrightarrow{AM}$
✓ uses scalar product
✓ simplifies scalar product
✓ uses magnitudes
✓ justifies conclusion

Question 19 **(8 marks)**

Functions f and g are defined as $f(x) = x^2 + ax + 2a$ and $g(x) = \frac{x}{x+b}$, where a and b are constants.

(a) Let $a = -4$ and $b = 3$.

(i) State, with reasons, whether the composite function $f(g(x))$ is a one-to-one function over its natural domain. (2 marks)

Solution
No because - composite function has two roots at $x \approx -3.7, -1.8$ - horizontal line cuts graph twice (with sketch from CAS) - etc
Specific behaviours
✓ reason ✓ support for reason

(ii) Determine any domain restrictions required so that the composite function $g(f(x))$ is defined. (3 marks)

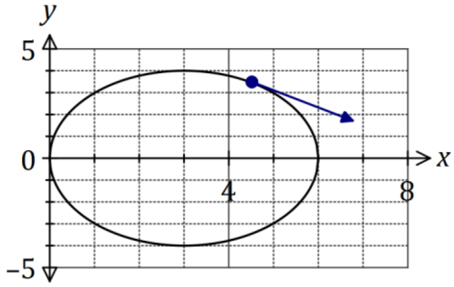
Solution
$$g(f(x)) = \frac{x^2 - 4x - 8}{x^2 - 4x - 5}$$ $$x^2 - 4x - 5 = (x+1)(x-5) \neq 0$$ $$x \neq -1, \qquad x \neq 5$$
Specific behaviours
✓ indicates composite function ✓ indicates denominator non-zero ✓ domain restrictions

(b) Determine the relationship between a and b so that the composite function $g(f(x))$ is always defined for $x \in \mathbb{R}$. (3 marks)

Solution
$$g(f(x)) = \frac{x^2 + ax + 2a}{x^2 + ax + 2a + b}$$ $$x^2 + ax + 2a + b \neq 0$$ $$a^2 - 4(2a + b) < 0$$ $$b > \frac{1}{4}a^2 - 2a$$
Specific behaviours
✓ indicates composite function denominator non-zero ✓ uses quadratic formula to create inequality ✓ states relationship

Question 20 **(8 marks)**

The position vector of a boat motoring on a lake is given by $\mathbf{r}(t) = (6 - 6\cos^2(t))\mathbf{i} + 4\sin(2t)\,\mathbf{j}$, where t is the time, in hours, after it leaves (0, 0) and distances are in kilometres. The path of the boat is shown below, where the shoreline is represented by the line $x = 0$.



Solution (b)
See graph
Specific behaviours
✓ position and direction

(a) Express the path of the particle as a Cartesian equation. (3 marks)

Solution
$6 - 6\cos^2(t) = 3 - 3\cos(2t)$ $\dfrac{y}{4} = \sin(2t)$ and $\dfrac{x-3}{-3} = \cos(2t)$ $\left(\dfrac{x-3}{3}\right)^2 + \left(\dfrac{y}{4}\right)^2 = 1$
Specific behaviours
✓ uses double angle identity   ✓ isolates $\sin(2t)$ and $\cos(2t)$ ✓ Cartesian equation

(b) On the graph above, mark the position of the boat when it is first 4.5 km from the shoreline and indicate the direction it is travelling. (1 mark)

(c) Determine the speed of the boat when it is first 4.5 km from the shoreline. (4 marks)

Solution
$6 - 6\cos^2(t) = 4.5 \Rightarrow t = \dfrac{\pi}{3}$ $\mathbf{v}(t) = 12\cos(t)\sin(t)\,\mathbf{i} + 8\cos(2t)\,\mathbf{j}$ $\mathbf{v}\left(\dfrac{\pi}{3}\right) = 3\sqrt{3}\mathbf{i} - 4\mathbf{j}$ $s = \sqrt{43} \approx 6.56\ \text{km/h}$
Specific behaviours
✓ determines time   ✓ obtains velocity vector   ✓ velocity vector at time   ✓ speed

Question 21 (7 marks)

(a) Consider the complex equation $z^5 = 4 - 4i$.

Solve the equation, giving all solutions in the form $r\operatorname{cis}\theta$ where $r > 0$ and $-\pi \le \theta \le \pi$.

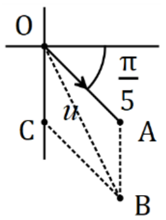
(4 marks)

Solution
$z^5 = 4\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$
$z_n = \sqrt{2}\operatorname{cis}\left(\frac{8k\pi - 17\pi}{20}\right), k = \{0, 1, 2, 3, 4\}$
$z_0 = \sqrt{2}\operatorname{cis}\left(-\frac{17\pi}{20}\right), z_1 = \sqrt{2}\operatorname{cis}\left(-\frac{9\pi}{20}\right), z_2 = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{20}\right), z_3 = \sqrt{2}\operatorname{cis}\left(\frac{7\pi}{20}\right), z_4 = \sqrt{2}\operatorname{cis}\left(\frac{15\pi}{20}\right)$
(Arguments in degrees: $-153°, -81°, -9°, 63°, 135°$)
Specific behaviours
✓ converts to polar form ✓ forms general expression for roots ✓ gives one correct root ✓ lists all correct roots

(b) One solution to the complex equation $z^5 = -9\sqrt{3}$ is $z = \sqrt{3}\operatorname{cis}\left(\frac{3\pi}{5}\right)$.

Let u be the solution to $z^5 = -9\sqrt{3}$ so that $-\frac{\pi}{2} \le \arg(u) \le 0$. Determine $\arg(u - \sqrt{3}i)$ in exact form.

(3 marks)

Solution	
 $OA = u, AB = -\sqrt{3}i$ $\lvert OA\rvert = \lvert OC\rvert = \lvert AB\rvert = \lvert BC\rvert = \sqrt{3}$	$\arg(u) = \frac{3\pi}{5} - 2\left(\frac{2\pi}{5}\right) = -\frac{\pi}{5}$ $\angle AOB = \frac{1}{2} \times \left(\frac{\pi}{2} - \frac{\pi}{5}\right) = \frac{3\pi}{20}$ $\arg(u - \sqrt{3}\,i) = -\frac{\pi}{5} - \frac{3\pi}{20} = -\frac{7\pi}{20}$
Specific behaviours	
✓ indicates argument of u ✓ uses geometric property of rhombus ✓ correct argument	

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________